MATH4581 Notes Typed

First lecture:

A Review of ordinary differential equations

And Power Series

Upon thinking about PDE’s….

A first order system of ODE’s can be written as:

$X'(t) = f(x(t),t)$

Where the unknown is this vector value function $x = [x1(t), x2(t), \dots xn(t)]$

And the domain t is real numbers

The codomain

$X:R \to R^n$ gives you values in R^n

The image is a set of points you ataully get back by giving points in a domain.

So if you have numbers on a real number line, you take the function X and it gives you this curve or path in R^n called $X(t)$

When the path crosses itself, the function is not 1- to -1

We have:

$F: R^n \times R \to R^n$

(the first part is the domain and the second part is the codomain)

Autonomous -- > stuff that is not explicitly time-dependent

Every ODE is equivalent to:

$X' = F(X,t)$

For example:

$Y'' + 3y' + y = 7$

The ;linear operator

$L[y] = y'' + 3y' + y$

What is the system version?

$X1 = y$

$X2 = y'$

$X = (x1; x2)$

$X' = (x2,$

$7-x1-3x2)$

Is this homogeneous?

No because it's equal to 7

It can look like this:

$X' = (0\ 1; -1\ -3) X + (0; 7)$

……

1) $X' = F(X,t)$
2) There is a general theory of ODE's

It's called the

EXISTENCE AND UNIQUENESS THEOREM

If you take any point Xo in R^n, there is a unique sokutuib, Recal……. (diagram on page 3

There is NO general theory of PDEs

SO WHAT CAN WE DO?

The three intro 2nd order linear constant coefficient PDE’s are:

Laplace’s Equation (elliptical)

Laplace (Triangle) = ux1x1 + ux2x2 + … + uxnxn = 0

Here, u: R^n __>)

-Heat Equation: ut = Laplacian of u

-Wave Equation:

Utt = Laplacian of u

Notation:

U is a domain in the subset of R^n

GRAPHS

-del/nabla and Laplacian

So you have some X on U that’s in R^n and a function u transforms it into some u(X) that is on the real number line R. A graph of a function of 2 variables is what’s called a surface in R^3.

G = { (x1, x2, u(x1,x2): (x1,x2) e U)

That means:

A set of points x1, x2, u(x1,x2)

Such that (x1,x2) is in U. If n>3, this is called a “hypersurface”.

X means you are using a vector (in this document).

*(****

Curvature

F’’(x) / [(1 + f’(x))^2]^3/2

;Laplacian of u = ux1x1 + ux2x2 = 0

This needs to be a saddle shape to cancel out to zero

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OPERATORS

The upside-down triangle is called the “nabla” – it’s the gradient: partial/partial, partial/partialx2, …, partial/partialxn)

It’s an operator.

Gradient of u = (partial-u/partial, partial u/ partial x2 etc)

It’s a vector

For gradient, you use “Du”

Also…if you do the dot product of the gradient and a vector, you get the divergence of that vector.

divV = dv1/dx1 _+ dv2/dx2 + … + dvn/dxn

It’s a sum.

The Laplacian of u is the dot product of two nablas to u. So, it’s the sum of second derivatives of a vector.

NEXT LECTURE

Review of Last Time

-Review of Power Series vs. Fourier Series

-1-D Heat Equation

Let’s skip curvature….. it’s on page 6

Let’s go onto POWER SERIES

We have e^x = the sum of 1/j! * x^j.

In general, f(x) kind of equals the sum of f^(j) * x0 / j! * (x-xo)^j

(as long as the derivates f’(xo), f’’(xo) etc ecist

There are some functions where the power series doesn’t work.

So for those we have the Fourier Series:

Instead of using polynomials, we can use sine waves

FOURIER SINE SERIES:

F(x) = sum of aj*sinjwx

But…since sine waves repeat, we need an interval. Otherwise we can't capture stuff like e^x

So let's try to do e^x. on an interval of interest: [0, L].

Meanwhile

d/dx sin(jwk) = jwcos(jwx)

d^2/dx*sin(jwx) = -(jw)^2sinjwx)d^2/dx2^2 [sin(jwx1)(sinjwx2)]

anyway if we had some kind of vector of a bunch of sin fxns,, for example:

[sinjwx1* sinjwx2 * … * sinjwxn] – let's call it B

(It's not a vector…it's a bunch of sine functinos multiplied together

And you take the Laplacian, it would just be this:

Laplacian of B = -n(jw)^2*B

OK CONTINUING WITH FOURIER SERIES

U(x,t) = e^-n(jw)^2*t * B, X = (x1, x2, … , xn)

-----du/dt = ut = -n(jw)^2u

This is kind of a pattern Fourier noticed

LaplacianB = -n(jw)^2B)

Spacial Laplacian

DELTAu = -n*(jw)^2 * u = ut

The FOURIER basis has solutions of the heat equations “built” into it

Solutions to the heat equation

N = 1

U(x,t) = e^-t*sinx

Satisfies du/st = s^2u/dx^2..

When t = 0

It’s a curve. As t in creases, the curve flattens.

The graph is on page 9.

Fourier sine series expansion STEP 1

X^x

Add periodic extension of e^x from [0, L]

At least we have a chance to represent this extension as a Fourier series.

Pick a w.

Sin(jwx)

It’lll be sin(Lx)

2pi / w = 2L.

Step 2:

Frequency is pi/L/

Sin(jpi/L*x)

This is an ODD periodic extension. Let’s call it g(x)

G(x) is about equal to the sum of ajsinjpix/L

STEP 3: Find the coefficients of aj

G(x) = sum aj* sin(jpix/L

You can multiply by one Fourier mode

And integrate both sides of the equation over the period.

Like this:

Int(from -L to L) of g(x) sin koiix/L*dx = sum(aj* int(from -L to L) of sin(jpix/L) * sin(kpix/L) *dx

This second term goes to zero. This is because unless k is equal to j, the integral of the sines multiplied by each other cancel out…..we are left with a sin^2(kpix/L)*dx.

Itengrated from -L to L./

This equals the same as cos^2(kpix/L) * dx

Integrated from -L to L. Becausse the graph of sines^2 and cos^2 is just shifted.

2L = 2*the integrate(from -L to L) of sin^2 (kpix/L) *dx.

SO

What do we have? We have that

\The integrate from -L to L of g(x)*sinkpix/L)*dx = sum(aj) * integral (from -L to L) of sinjpix*sinkpix/L *dx

= akL

Ak = 1/L * integral (-L to L of g(x) * sin (kpi/L)dx

Ak = 2/L * integral (0 L* e^x * sin(kpix/L) dx

SO this is our coefficient

SO

E^x MAY equal the sum of ……

Well anyway

Next lecture

Fourier know that e^-t*sinx was a solution to the heat eq.

Initial condition was u(x,0) = sinx. If the initial condition is a trig function, then solution is known.

Also (u(x,t) = e^-j^2t * sinjx

ALSO works.

The purpose of the Fourier series is that u(x,0) could be anything.

Say we want to model the conduction of heat energy in a thin rod.

The 1-D heat equation as one spatial dimension)

Thermal energy density, theta, is a function of position and time

If you integrate over space, you get total energy.

Defining power of theta is that the total thermal energy in a rod is

Total energy = integral (from 0 to L) of theta(x,t)*dx.

If you put a dummy variable a in between x and x + dx, you can get a teeny tiny bit of thermal energy de, which is the integral ofrom x to x + dx of theta(a,t)*da.

It's a function of time…so add a time derivatve?

d/dt * the integratl from x to x+ dx of theta (a,t) * da

Thisi s the rate at which thermal energy enters the region (interval)

HOWS iT GETTING IN THERE, THOUGH?

The rod only conducts each through the ends.

Physical dimensions of theta? Heat energy density?

Hint: velocity…. Length / time

[energy] = [force] * L

= mass*Length / Time^2 * Length

So Theta = [energy] / L = Mass* L / T^2

Ok? Ok!

Lineal heat flux: phi = phi(x,t)

It’s the rate at which heat energy is crossing x.

Phi is the rate at which thermal energy moves to the right at x.

(integral is on page 16)

It’s the integral from x to x + delta x of theta of (xi,t)*dxi and that is equal to the heat flux function phi(x,t) minus phi(x + delta x, t)

When we model heat conduction, we get this.

Not that the delta x is positive here.

What can we get from this?

If you put a minus sign (basically take -1 out of the RHS of the equation and differentiate under the integral you get the integral form of the heat equation.

(That is on page 16)

Also you get that it equals zero.

Iso now you can use the Fundamental Lemma of Vanishing Integrals!

Which states:

If F is a continuous function with the integral of f over a region equaling 0 for all regions R, then f(x) = 0 for all x.

Because of this Lemma, we have this pointwise form equation:

Dtheta/dt = - dphi/dt.

“The time derivative of heat energy density is equal to the negative of the special derivative of the flux.”

More general:

External sources/sinks of thermal energy come in the form of Q

Q is defined by:

The integral from x to x + delta x of Q * dxi. That's the rate of "other" heat energy change in the interval.

[Q] = [energy] / Length* Time

The integral form of the heat equation is now having a Q term in it.

So the poinwise form is:

Dtheta/dt = -dphi/dx + Q.

Las function…. (dunno what "last function mmmeans"

You have some parameters,: theta, phi, and Q. Q is sometimes known. But you need this in terms of something that's easy to measure: temperature! This is in terms of u.

Assumptions:

-Law of specific heat says that:

Theta = c*rho*u

Where c is the specific heat capacity, rho is mass density, and u is the temperature.

-Fourier's Law:

Phi = -K*u_x

Which means that flux at a point is the temperature gradient. K is the constant of thermal conductivity.

(Integral form and pointwise form are on page 18)

Other forms, if c, rho, and K are constant:

U_t = k/crho * u_{xx} + Q/crho.

k/rhoc is the diffusivity constant, k. And the Q part is sometimes just f.

So we get:

$U_t = ku_{xx} + f$.

The hgeat equation!!!!

Or we have it in these weird terms:

Lu = ut-kuxx (Linear partial differential operator)

Lu = f. L is the “heat operator” and f is the “forcing function”. Which is proportional to the source Q/crho.

If we zero out f, we get a homogeneous, constant thermal properties.

If the diffusivity k =1 1, we get this niece PDE. U+t = u_{xx}

It’s a parabolic PDE

U_{xx} is the Laplacian of u in one dimension.

NOTE

Differentiation under an integral nees continuous 1st partials.

One more thing:

The solution u(x,t) = e^-tsinx

How fast is the heat going out at each end? Fourier's Law.

A NEW DAY Sep 2 page 21

What are the boundary conditions for insulated ends?

1-D. Fourier's Law: phi(x,t) = $-ku_x(x,t)$

If there's no phi at the endpoints, we get the boundary condition:>

Ux(0,t) == 0

And

Ux(end, t) = 0

Take this ant the heat equation, and you can claim that as t goes to infinity that the function will hit 1/L * integral from 0 to L of the function u0(x)*dx.

Regularizing (page 22)

Parabolic PDE are instantaneously regularizing.

Infinite propagation speed

Parabolic equations can "move things" infinitely fast.

Regularized means that derivatives exist at all orders.

In the

I don't really understand this so much….but it's in page 22.

MOVING ON

Heat equation in other dimsnions:

Start with energy density:

Thera[theta] = [energy]/L^3 this is volumetric energy density.

-New ideas for N>1

-Integration on sets

-calculating Reimann sums of little pieces dxi

Multiply by the dimension of the pieces

Thnen you get

The integral of theta over a region U is equal to the integral of theta over x which is in the region U.. X is a point that moves around U. The dimensions of U are n.

- ➔ Fourier's Law is now
- ➔ Phi(vector) = ;KDu (where D is the spatial gradient)
- ➔ →Fundamental theorem of Calculus in higher dimensions ---- DON'T KNOW THAT YET

Fundamental theorem of calc in 2D is that…

a) The integral of f'(x)dx from a to b is f(b) – f(a).
b) d/dx * the integrat from a to x of g(xi) dx = g(X)/.

This is used on the 1D hgeat equation.

d/dt*the integral of theta over region U = phi(b) – phi(a)

But in R^n,

d/dtintegral over U of theta is equal to the negative flux integral dotted with the normal vector over the boundary of U. (This is page 24)

This allows you to computer rates.

The replacement of the Fundamental Theoreum of Calculus is the DIVERGENCE THEOREM.

The divergence theoreum is that the divergience of a vector field v over the region U is equal to the vector field dotted with the normal vector over the boundary of the region U.

YEE!

The proof of it is on page 25.(dunno where this proof was going)

Separation of variables examples are on page 33.and end on page 38!!

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A NEW DAY page 39

Let’s talk about TRANSPORT equations

What does transport mean?

The medium itself is moving around.

Transport mass is a good example

Start with a vector field v, [v] = Length / Time

Let’s take a volumetric mass density rho,

Rho = M/Length ^3

So

Rhov = M/T*L^2 … mass per time on n-1 space dimensions.

These are dimensions of a…..FLUX

You can integrate over a lower dimension to get mass pertime.

TRANSPORT EQUATIONS CONTD

It’s the rate at ewhich mass is transported across a surface.

Delta is the surface…. Integral over d of rho*v dotted with n.

In particular, it’s rho*v dotted with n integrated over the boundary of U.

* * *

So, if you have mass conservation, and spatially dependent mass density

Rho = rho(x,t)

Then the rate at which mass enters R

d/dt integral over R of rho is equatl to the negative of rho*v dotted with n over the boundary of R. Use the divergence theorem and you get:

rho_t = -div(rhov)

This is the continuity equation for fluid mechanics.

* * *

What is the divergence of a vector field?

The bad answer is that it’s a bunch of partials added u.

The good answer is that if you look at the integral ov v dotted with n over boundary R, inmagine that the limit as R approaches some point P in the middle of R of the integral of the vector v dotted with n over the boundary R is equal to 0.

(this is on page 41)

We get 0/0….

So let’s try a different limit

…..eh I don't really understand

"PROOF" of the DIVERGENCE THEOREM

If you cut up U into little reions Rj that are not overlapping, you get that the integral of vectors dotted with n over the boundary of U is equal to the sunm of vectors dotted with n over the boundarys R.

Do the reimann sum for volumes and you end up with the divergence. BINGO BANGO.

INTRINSIC MATHEMATICS

Herman Weyl: "The introduction of numbers as coordinates is an act of violence"

The intrinsic divergence is that the limit of a vector field dotted with normals integrated over the boundary R times 1/muof R as epsilon and delta approach 0 is the divergence.

What about the intrinsic Laplacian?

For n = 2, the Laplacian is equal to the divergence of the gradient of u.

To get an intrinsic Laplacian, you need an intrinsic gradient.

Affine polynomial in x.

Claim: l(x) = mx + b is not linear unless b = 0. Proof if l is linear in x, then l(x1 + x2) = l(x1) + l(x2)

Mx1 + mx2 + b does NOT equal mx1 + mx2 + 2b unless b =0.

I WAS LATE ON THE DAY THEY DID HEAT CONDUCTION IN A RING OMG that is on page 49.

Laplace's Equation

Equillibrium solutions of $u_t = u_{xx}$ all satisfy Laplacian of u = 0.

So that means the separation of variables can work sometimes.

(Solutions on Page 53)

It ends on page 55

Calculus of Variations page 56

All you need to know is the first order necessary condition for an extrema

-This is about finding minima of real valued functions whose domain is an infinite dimensional set of functions.

These start on page 56.

(If you have a max/min, your derivative is 0.

Infinite dimensional set of functions:

Example:

Fund the set of all functions u such that u(x) = u, u(x2) = uu2

Prescribed enpoints

U could also be a NUMBER, like, a length.

L[u] = the entegral from x1 to x2 of the quare root of 1 + u'(x)^2 * dx

i.e., the length "functional".

In order to find the shortest distance between the 2 points, you take the "first variation" and find the minimum. You're _____ (dunno what this was supposed to be)

Then there is

D[u] = int3egral from x1 to x2 u’(x)^2 dx …. “penalized for u’(x) being non-zero

This is Dirchlet “energy”

N=2 (or higher)

U is a region in R^2

You have a set of u : u across the boundary is prescribed) = g

D[u] = integral along the region U of the modulus of the gradient of u squared.

This in R^2 is the integral along the region U of du.dx^@ + du./dy^2.

The domain of Dirichlet energy:

Admissable class (set of all functions U such that the boundary of u ……..

Basically u has to hit the function g when it gets to the boundary. (There is a good picture of this on page 58.

D[u] = the inetegral over U of the modulus of the gradient of u squared.

But….the Area [u] = the integral over the region U of the square root of 1 + |Du|^2 will have different minimizers.

D[u*] < or = D[u] for all u in A (the admissible class)

Just like at the minimum, f(x*) is less than or equal to f(x) for all x between a and b.

(As an aside)

The length of a tangent segment is the sum of1 + u’^2 (there is a nice pic of this on page 59

OKAY so we want to find the minimizer u* of D[u] = Region integral |Du|^2.

BIG QUESTION IS: What is the derivative of D at u*?

What we can do is look at “competitors.

A competitor function is u = u* + ephi

This is a variation of u.

Phi is some “bump” function. E can vary and is a scalar.

This is called “wiggling the minimizer.”

D[u* + ephi] >= D[u*] So, the competitor function always has to be bigger than the minimum.

Phi is fixed. U* is fixed, therefore the only thing moving is e.

D[u* + ephi] >= D[u*]

LHS is a f€ and the right hand side, let’s say is f(0). Because f’0) = 0…around zero, e mist be ghigher.

Then we get the inequality to be

F(E) >= f(0).

So, we need to evaluate the derivative of the LHS of the equality for when e = 0.

The derivative is like this:

d/de of D [u* + ephi] | e = 0

VThis is a first variation.

It looks like a lower case delta.

New notation

delta*Du*[phi] = d/de D[u* + ephi]

=**The first variation of the Dirichlet energy at u* in the direction phi. (This is on page 60)**

Compute

D[u + ephi] = region integral U |D(u + ephi)|^2

Since gradient is a linear fxn, we can do region integral of |Du + eDphi|^2

= Region integral [|Du|^2 + 2eDu dot Dphi + e^2|Dphi|^2]

*We can differentiate under the integral sign because the limits of integration have nothing to do with the boundary U

dDu[phi] = d/deD[u + ephi] | e = 0

d/deD[u+ephi] = region int [2Du dot Dphi + 2e|Dphi|^2]

e = 0, so dDu[phi] = 2*region int (Du dor Dphi

For a minimizer u*

Region int Du* dot Dphi = 0 for all directions phi.

WHAT DOES TYHIS MEAN?

We can't vary phi too much around the baoundary values.

A (Admissable class) = u: u.|du = g}

Therefore u + ephi, phi|dU = 0

Basically the bumps are well inside the boundary.

There is a sret of x where phi(x) is not equal to zero. Called the supporting phi.

For a minimizer u* of Dirichlet energy, we havThe region integral of Du dor Dphi = 0 for AL:L phi

In the big scheme of things, we should be thinking, what does this tell me about u*?

The DOIVERGENCE THEOREM = div (phiDu) = Dphi dot Du + phidiv(Du)

Therefore the region integral of Du dor Dphi = -integral(div Duphi) + region integral div phiDu

That last term vanishes so…

=- region int (divDu) phi

And this vanishes for every phi.

This means that

Minimizers of Dirichlet are solutions of Laplace's equation!!!!!

LAST TIME

(this is page 63)

Admissible class

A is in the set u such that the boundary of u is a function g}

Problem: minimize F:A _> by

F[u] = integral (F (X,U, Du)

Lagrangian integral functional. The integrand is a lagrangian.

2n + 1 variables u:U _> R

Strategy: minimize F[u] = integral F(x,u,Du)

Over u in A

Strategy: use first variation

(1st order necessary condition_

V ← class of admissiole purturbations

Fix phi in V, u + ephi in A.

Assume that u beats all the competitors

So

dF[phi] = d/de F[u + ephi] |e = 0 = 0

A more generalized version of what we had last time.

Also thigs:

d/de F [u +ephi] |e = 0 = 0 for all phi in V

So the d/de means we need to differentiate with respect to e.. u + ephi cold be a vector.

So we have three slots in the Lagrangian integrand.

F = F(x, z, P)

Z is a function, z is one real slot ofr a function, and P is a gradient slot…could be a vector

Start by differentiateion under the intrgral + chain rule.

(This is all on page 64….it's really hard to write this down.)

Fundamental Lemma of the Calculus of Variations:

If f is continuous and f dot phi integrated over U = 0 for all phi with support phi =< u,, then f(x) = 0 for all x in U.

Page 69 has the Euler-Langrange PDE for F (fancy F)

Proof that Laplacian sol are Dirichlet energy